A Conversation on Division by Fractions

This chapter has two aims in mind: providing another opportunity for the teaching of division by fractions, and illustrating a few teaching principles. The explanation of the rule of division by a fraction is probably hard for children. Here is a way which is more suited to being taught in class. It is given in a form of a conversation with my daughter, Geffen. When she was in 4th grade, I took her for a walk. During that walk, which lasted less than an hour, I taught her how to divide a number by a fraction. This conversation is reproduced here quite accurately, and is used also as an opportunity for illustrating a few teaching principles. The principles involved in each stage appear parenthesized and italicized.

I: We have learned how to multiply by a fraction. Let us now see how to divide by a fraction. For example, how to calculate $10 \div \frac{2}{3}$. How do you think we should start?

G: We should ask a simpler question. (*Start from the simplest question possible. Also: Share with the student the principles of sound thinking. The rule of starting from the simple should be explained to the students.*)

I: Right. Let us start then with dividing by a simpler fraction. What is the simplest fraction you can think of?

G: $\frac{1}{2}$.

I: Indeed, $\frac{1}{2}$ is a fraction we know and understand well. What is the simplest exercise you can think of in which we divide by $\frac{1}{2}$?

G: $1 \div \frac{1}{2}$. (*Let the student invent the problems.*)

I: Can you calculate that?

G: Yes, it is 2, since $1 \div 2 = \frac{1}{2}$. If $6 \div 2 = 3$ then $6 \div 3 = 2$, and likewise if $1 \div 2 = \frac{1}{2}$ then $1 \div \frac{1}{2} = 2$. (*Well, this is smart. Not every child would see that. But what follows does not necessitate such insight.*)

I: Very nice! But here is another way, which I myself understand better. $6 \div 2 = 3$ because 2 goes into 6 three times. Do you remember what we call such division?

G: Yes, containment division. (*Use precise words, and distinguish fine points of meaning.*)

I: If $6 \div 2$ means how many times 2 goes into 6, what does $1 \div \frac{1}{2}$ mean?

G: How many times does $\frac{1}{2}$ go into 1?

I: And how many indeed?

G: $\frac{1}{2}$ goes into 1 two times. So $1 \div \frac{1}{2}$ is 2. (*Inadvertently, we followed here another teaching principle: Try to see the same thing from as many viewpoints as possible. We saw two ways of calculating* $1 \div \frac{1}{2}$.)

I: Could you tell me now what is $3 \div \frac{1}{2}$? (*Add one ingredient at a time.*)

G: Yes. $\frac{1}{2}$ goes 2 times into 1. Three is 3 ones, so $\frac{1}{2}$ goes $3 \times 2 = 6$ times into 3. So $3 \div \frac{1}{2}$ is 6.

I: Right. And $4 \div \frac{1}{2}$?

G: 8.

I: And $5 \div \frac{1}{2}$? (*Stabilizing the knowledge, by exercising.*)

G: 10.

I: Could you tell me the rule?

G: Yes, dividing a number by $\frac{1}{2}$ multiplies it by 2. Because each 1 in the number contains 2 halves. (*After experiencing a rule, formulate it in words.*)

Remark: In class, extensive exercising is needed at this point. How many halves of an apple are there in 5 apples? What is $5 \div \frac{1}{2}$? How many times does $\frac{1}{2}$ go into 10? What is $10 \div \frac{1}{2}$? 13 chocolate bars were divided between children, and each got half a bar. How many children were there? What is $13 \div \frac{1}{2}$?

I: Fine. Let us now divide by a third. What is $1 \div \frac{1}{3}$?

G: $\frac{1}{3}$ goes into 1 three times, so it is 3.

I: And what about $4 \div \frac{1}{3}$?

G: $\frac{1}{3}$ goes 3 times into 1, so it goes $4 \times 3 = 12$ times into 4.

I: What is the rule for dividing by $\frac{1}{3}$?

G: Dividing a number by $\frac{1}{3}$ is multiplying it by 3.

I: Great. So, we know that dividing by $\frac{1}{2}$ is multiplying by 2, and dividing by $\frac{1}{3}$ is done by multiplying by 3. How do you divide by $\frac{1}{4}$?

G: By multiplying by 4.

I: Yes, you've got the principle. Dividing by one over a number is multiplying by the number (*This would be too hard for her to formulate.*) Let us now divide by $\frac{2}{3}$, which is what we started with. Let us first return for a moment to division by $\frac{1}{3}$. If in a party there were 10 cakes, and every child got a $\frac{1}{3}$ of a cake, how many kids were there?

G: $10 \div \frac{1}{3} = 30$; there were 30 kids.

I: And suppose now that each kid gets $\frac{2}{3}$ of a cake, instead of $\frac{1}{3}$, that is, 2 times more than before. For how many kids will the cake suffice?

G: Every kid now gets what 2 kids got before. So, there will be half of 30 kids, which is 15.

I: Right. What exercise did you do here?

G: $10 \div \frac{2}{3}$, because we found how many times $\frac{2}{3}$ goes into 10.

I: Right, and what operations did you do?

G: I multiplied the 10 by 3, and divided by 2.

I: So, what is the rule for dividing a number by $\frac{2}{3}$?

G: You take the number, multiply it by 3 and divide by 2.

I: And what would be the rule for dividing by $\frac{3}{4}$?

G: Multiply by 4, and divide by 3.

I: Very nice. And what is the rule for dividing by a general fraction?

G: You multiply by the denominator, and divide by the numerator.