What is Mathematics?

The book of nature is written in the language of mathematics.
Johannes Kepler, Astronomer

The Queen of Sciences

Mathematics is the queen of sciences and arithmetic the queen of mathematics.
Carl Friedrich Gauss, Mathematician

In second grade classes, I try to explain the importance of numbers. I tell the children a story of a king who hated numbers so much that he forbade their use throughout his kingdom. Together we try to imagine a world free of numbers, and discover that life in it is very limited. Since it is forbidden to mention a child's age, children of all ages enter the first grade; you cannot pay for your groceries, nor can you set up an appointment, since you are not allowed to mention the number of hours and minutes.

This is only an illustration of the importance of mathematics in our lives. As civilization and technology advance, our lives become more and more dependent on mathematics. Steven Weinberg, a Nobel laureate in physics, dedicated two chapters in his book, "Dreams of a Final Theory," to subjects beyond physics: mathematics and philosophy. He writes that time and again he is surprised to discover how useful mathematics is, and how futile is philosophy.

To understand why this is so, one must understand what is mathematics. This is not a simple question — even professional mathematicians find it difficult to answer. Bertrand Russell said of mathematicians that they "don't know what they are doing." (His judgment of philosophers was even harsher: A philosopher in his eyes is "a blind man in a dark room looking for a black cat that isn't there.") This is true in at least one sense: Most mathematicians do not bother to ask themselves what it is, exactly, that they are doing.

To try answering this question, we will start with a simple example: What is the meaning of $3 + 2 = 5$?

In the first grade, I ask the children to examine how many pencils there are when you add 3 pencils to 2 pencils. They know that

"addition" means "joining." Therefore they join 3 pencils and 2 pencils and count: 5 pencils. Now I ask, "How many buttons are there when you add 3 buttons and 2 buttons?" "Five buttons," they answer, without missing a beat. "How do you know?" I insist. "We know from the previous question." "But the previous question was about pencils. Maybe it's different with buttons?"

They laugh. But not because the question is pointless. On the contrary. It contains the secret of mathematics: abstraction. It does not matter if the objects in question are pencils, buttons or apples. The answer is the same in all cases. This is why we can abstractly say: $3 + 2 = 5$.

This is an elementary, but characteristic, example: Mathematics abstracts thinking processes. Obviously, every thought is abstract to a certain degree. But mathematics is unique in that it abstracts the most elementary processes of thought. In the example of $3 + 2 = 5$, the process involved is the joining of objects — 3 objects and 2 objects. One can ask many questions about these objects: What are they — pencils or apples? Are they in your hand or on a table? And if they are on a table, how are they arranged? Mathematics ignores all these details, and asks a question that relates not to the various details, but only to the fact that these are objects that are joined: the resulting amount. That is, how many objects are there?

Abstract thought is the secret of man's domination of his environment. The power of abstractions lies in the fact that they enable us to cope efficiently with the world. In other words, they save effort. They enable going beyond the boundaries of the "here and now" — something discovered here and now can be used in another place and at another time. If 3 pencils and 2 pencils equal 5 pencils, the same will be true for apples, and it will also be true tomorrow. A one-time effort provides information about an entire world.

If abstractions in general are useful, then all the more so is mathematics, which takes abstractions to their limit. Therefore, it is not surprising that mathematics is so useful and practical.

Should Everyone Learn Mathematics?

People, on learning that I am a mathematician, often react with a thin smile, barely hiding a grimace of agony: "Mathematics wasn't one of my strong subjects." For so many people learning mathematics is such a tormenting experience, that each generation asks the same question — what for? Why is this torture necessary? Shouldn't most people just give up on the attempt to learn mathe-

matics? Nowadays, when a calculator can instantly perform mathematical operations — what is the point of learning the multiplication table, or long division?

One answer is that mathematics is the key to all professions demanding knowledge of the exact sciences, and there are many of those these days. But mathematics is important not only for understanding reality. It offers much more than that — it teaches abstract thought, in an accurate and orderly way. It promotes basic habits of thought, such as the ability to distinguish between the essential and the inessential, and the ability to reach logical conclusions. These are some of the most significant assets that schooling can provide.

The question still remains unanswered — why is it so difficult? Must mathematics be a cause of suffering? A currently popular answer is "no" — the problem lies in the teaching. Common opinion is that many children considered to be "learning disabled" are actually "teaching disabled." But it can't be that simple. Blaming the teachers is too simplistic, and unreasonable. Anyone who claims that for hundreds and thousands of years mathematics teachers have been doing a bad job, must explain why this is so — and why it isn't so in other subjects.

The special problem in teaching mathematics lies in the difficulty of conveying abstractions. You can tell people the name of the capital of Chile, but you can't abstract for them. *This is a process each person must accomplish on his or her own.* One must mentally pass through all the stages from the concrete to the abstract. The teacher's role in this process is to guide the student, so that he experiments with the principles in the correct order. This is not a simple art, and it is not easy to come by. But neither is it impossible. One of the purposes of this book is to relay some of the principles along the path of such "midwifery" teaching, as Socrates put it.